Optimization problems

Skills:

Given a function $f(x)$,

$f'(x) = 0 \longrightarrow x_0 = c$, critical value

- Sign of $f'(x)$ tells whether or not $(c, f(c))$ is min or max value of f
- second derivative test: $\begin{cases} f''(x_0) > 0 \Longrightarrow x_0 = c \text{ is min} \\ f''(x_0) < 0 \Longrightarrow x_0 = c \text{ is max} \end{cases}$

Example

Find two numbers such that their product is 80 and the sum of the first and two times the second number is minimum

Guess!

prod: $10 \cdot 8$

sum: $10 + 2(8) = \underline{26}$

prod: $5 \cdot 16$

Sum: $16 + 2(5) = \underline{26}$

$2 \cdot (40) = 80$

$40 + 2(2) = \underline{44}$

Mathematically:

Let x, y the two numbers:

prod: $xy = 80$ (a)

sum $x + 2y$ is minimized!

Let $f(x,y) = x + 2y$ to minimize

using (a), $xy = 80 \rightarrow y = \frac{80}{x}$

$\Longrightarrow f(x) = x + 2\left(\frac{80}{x}\right)$ to minimize!

$\Longrightarrow f'(x) = 1 - \frac{160}{x^2} = 0$ to find the critical value!

$\Longrightarrow x^2 = 160$

$\Longrightarrow x = \pm\sqrt{160} = \pm 4\sqrt{10}$

Verification: $f''(x) = \frac{320}{x^3} \longrightarrow f''(4\sqrt{10}) > 0$ so $4\sqrt{10}$ is the minimum value.

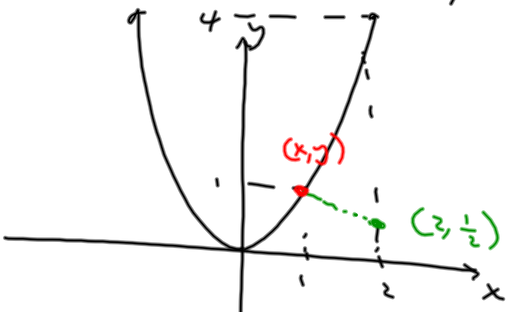
Now, $y = \frac{80}{x} \longrightarrow \frac{80}{4\sqrt{10}} = \frac{20}{\sqrt{10}} = 2\sqrt{10}$

The numbers are: $4\sqrt{10}$ and $2\sqrt{10}$

Check: product: 80

$4\sqrt{10} + 2(2\sqrt{10}) = 8\sqrt{10} \simeq 25.3$ a little better than our guess (26!)

Ex: Find the point on the graph of $f(x) = x^2$ that is closest to $(2, \frac{1}{2})$



goal: is to minimize the distance between $(2, \frac{1}{2})$ and (x, y)

distance $D = \sqrt{(x-2)^2 + (y - \frac{1}{2})^2}$

to mimize D is to minimize $(x-2)^2 + (y - \frac{1}{2})^2$

let $f(x,y) = (x-2)^2 + (y - \frac{1}{2})^2 \longrightarrow$

$$f(x) = (x-2)^2 + \left(x^2 - \frac{1}{2}\right)^2 = x^2 - 4x + 4 + x^4 - x^2 + \frac{1}{4}$$

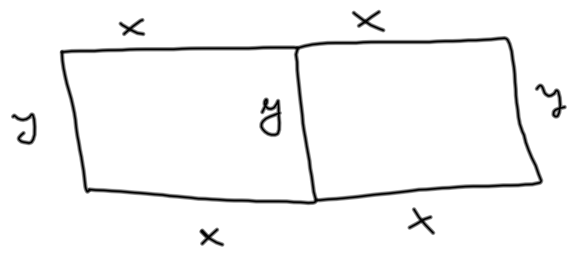
$$= x^4 - 4x + \frac{17}{4}$$

So $f'(x) = 4x^3 - 4 = 0 \longrightarrow x^3 = +1 \longrightarrow x = \sqrt[3]{1} = 1$

Check: $f''(x) = 12x^2$ at $x = 1 \longrightarrow f''(1) = 12$ so indeed $f(x)$ attains its minimum value when $x = 1$

$\Longrightarrow y = x^2 = 1$ Thus, the point $\boxed{(1,1)}$

Ex: A rancher has 560 ft of fencing to enclose two adjacent rectangular corrals. What dimensions should be used so that the enclosed area is maximum? (see figure)



Area: $A = 2xy$ is to be maximized subject to Perimeter: $4x+3y = 560$

$f(x,y) = 2xy$ since $y = \frac{560-4x}{3}$

$\Rightarrow f(x) = 2x\left(\frac{560-4x}{3}\right) = \frac{1120x - 8x^2}{3}$

so $f'(x) = \frac{1}{3}\left(1120 - 16x\right) = 0$

$\Rightarrow x = \frac{1120}{16} = 70$

Check: $f''(x) = -16 < 0$ maximum at $x = 70$

Now $y = \frac{560-4(70)}{3} = \frac{280}{3}$

Hence the dimensions length: $\frac{280}{3}$; width: 70